Generalized Measures of Edge Fault Tolerance in (n,k)-star Graphs*

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Abstract

This paper considers a kind of generalized measure $\lambda_s^{(h)}$ of fault tolerance in the (n,k)-star graph $S_{n,k}$ for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$, and determines $\lambda_s^{(h)}(S_{n,k}) = \min\{(n-h-1)(h+1), (n-k+1)(k-1)\}$, which implies that at least $\min\{(n-k+1)(k-1), (n-h-1)(h+1)\}$ edges of $S_{n,k}$ have to remove to get a disconnected graph that contains no vertices of degree less than h. This result shows that the (n,k)-star graph is robust when it is used to model the topological structure of a large-scale parallel processing system.

Keywords: Combinatorics, fault-tolerant analysis, (n,k)-star graphs, edge-connectivity, h-super edge-connectivity

1 Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network. The connectivity $\lambda(G)$ of a graph G is an important measurement for fault-tolerance of the network, and the larger $\lambda(G)$ is, the more reliable the network is.

A subset of vertices B of a connected graph G is called a *edge-cut* if $G-B$ is disconnected. The *edge connectivity* $\lambda(G)$ of G is defined as the minimum cardinality over all edge-cuts of G. Because λ has many shortcomings, one proposes the concept of the h-super edge connectivity of G, which can measure fault tolerance of an interconnection network more accurately than the classical connectivity λ.

A subset of vertices B of a connected graph G is called an *h-super edge-cut*, or *h-edge-cut* for short, if $G-B$ is disconnected and has the minimum degree at least h. The *h-super edge-connectivity* of G, denoted by $\lambda_s^{(h)}(G)$, is defined as the minimum cardinality over all h-edge-cuts of G. It is clear that, if $\lambda_s^{(h)}(G)$ exists, then

$$\lambda(G) = \lambda_s^{(0)}(G) \leqslant \lambda_s^{(1)}(G) \leqslant \lambda_s^{(2)}(G) \leqslant \cdots \leqslant \lambda_s^{(h-1)}(G) \leqslant \lambda_s^{(h)}(G).$$

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For any graph G and integer h, determining $\lambda_s^{(h)}(G)$ is quite difficult. In fact, the existence of $\lambda_s^{(h)}(G)$ is an open problem so far when $h \geqslant 1$. Some results have been obtained on $\lambda_s^{(h)}$ for particular classes of graphs and small h's (see Section 16.7 in [5]).

This paper is concerned about $\lambda_s^{(h)}$ for the (n,k)-star graph $S_{n,k}$. In h-super connectivity, several authors have done some work. For $k = n-1$, $S_{n,n-1}$ is isomorphic to a star graph S_n. Akers and Krishnamurthy [1] determined $\lambda(S_n) = n-1$ for $n \geqslant 2$ and $\lambda_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$. In this paper, we show the following result.

Theorem 1.1 *If $2 \leqslant k \leq n-1$ and $0 \leqslant h \leqslant n-k$, then*

$$\lambda_s^{(h)}(S_{n,k}) = \begin{cases} (n-h-1)(h+1) & for\ h \leqslant k-2\ and\ h \leqslant \frac{n}{2}-1, \\ (n-k+1)(k-1) & otherwise. \end{cases}$$

This result implies that at least $\min\{(n-k+1)(k-1), (n-h-1)(h+1)\}$ edges of $S_{n,k}$ have to remove to get a disconnected graph that contains no vertices of degree less than h. The proof of this result is in Section 3. In Section 2, we recall the structure of $S_{n,k}$ and some lemmas used in our proofs.

2 Definitions and lemmas

For given integer n and k with $1 \leqslant k \leqslant n-1$, let $I_n = \{1, 2, \ldots, n\}$ and $P(n,k) = \{p_1p_2\ldots p_k : \ p_i \in I_n, p_i \neq p_j, 1 \leqslant i \neq j \leqslant k\}$, the set of k-permutations on I_n. Clearly, $|P(n,k)| = n!/(n-k)!$.

Definition 2.1 *The (n,k)-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$. The adjacency is defined as follows: a vertex $p = p_1p_2\ldots p_i\ldots p_k$ is adjacent to a vertex*
(a) $p_ip_2\cdots p_{i-1}p_1p_{i+1}\cdots p_k$, where $2 \leqslant i \leqslant k$ (swap p_1 with p_i).
(b) $\alpha p_2p_3\cdots p_k$, where $\alpha \in I_n \setminus \{p_i : \ 1 \leqslant i \leqslant k\}$ (replace p_1 by α).

The vertices of type (a) are referred to as *swap-neighbors* of p and the edges between them are referred to as *swap-edge* or *i-edges*. The vertices of type (b) are referred to as *unswap-neighbors* of p and the edges between them are referred to as *unswap-edges*. Clearly, every vertex in $S_{n,k}$ has $k-1$ swap-neighbors and $n-k$ unswap-neighbors. Usually, if $x = p_1p_2\ldots p_k$ is a vertex in $S_{n,k}$, we call p_i the *i-th bit* for each $i \in I_k$.

The (n,k)-star graph $S_{n,k}$ is proposed by Chiang and Chen [4]. Some nice properties of $S_{n,k}$ are compiled by Cheng and Lipman (see Theorem 1 in [2]).

Lemma 2.2 *$S_{n,k}$ is $(n-1)$-regular $(n-1)$-connected.*

Lemma 2.3 *For any $\alpha = p_1p_2\cdots p_{k-1} \in P(n,k-1)$ $(k \geqslant 2)$, let $V_\alpha = \{p\alpha : \ p \in I_n \setminus \{p_i : i \in I_{k-1}\}\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph of order $n-k+1$, denoted by K^α_{n-k+1}.*

Let $S^{t:i}_{n-1,k-1}$ denote a subgraph of $S_{n,k}$ induced by vertices with the t-th bit i for $2 \leqslant t \leqslant k$. The following lemma is a slight modification of the result of Chiang and Chen [4].

Lemma 2.4 *For a fixed integer t with $2 \leqslant t \leqslant k$, $S_{n,k}$ can be decomposed into n subgraphs $S^{t:i}_{n-1,k-1}$, which is isomorphic to $S_{n-1,k-1}$, for each $i \in I_n$. Moreover, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{t:i}_{n-1,k-1}$ and $S^{t:j}_{n-1,k-1}$ for any $i, j \in I_n$ with $i \neq j$.*

Since $S_{n,1} \cong K_n$, we only consider the case of $k \geqslant 2$ in the following discussion.

Lemma 2.5 *If $2 \leqslant k \leq n-1$ and $0 \leqslant h \leqslant n-k$, then*

$$\lambda_s^{(h)}(S_{n,k}) \leqslant \begin{cases} (n-h-1)(h+1) & \text{for } \ h \leqslant \frac{n}{2}-1, \\ (n-k+1)(k-1) & \text{otherwise.} \end{cases}$$

Proof. By our hypothesis of $h \leqslant n-k$, for any $\alpha \in P(n, k-1)$, we can choose a subset $X \subseteq V(K^{\alpha}_{n-k+1})$ such that $|X| = h+1$. Then the subgraph of K^{α}_{n-k+1} induced by X is a complete graph K_{h+1}. Let B be the set of incident edges with and not within X. Since $S_{n,k}$ is $(n-1)$-regular and K_{h+1} is h-regular, we have that

$$|B| = (n-h-1)(h+1).$$

Clearly, B is an edge-cut of $S_{n,k}$. Let x be any vertex in$S_{n,k} - X$, and $d(x)$ denote the number of edges incident with x in $S_{n,k} - X$. In order to prove that B is an h-edge-cut, we only need to show $d(x) \geqslant h$. Note that X is contained in $S^i_{n-1,k-1}$ and edges between $S^i_{n-1,k-1}$ and $S^j_{n-1,k-1}$ are independent for any $i, j \in I_n$ with $i \neq j$ by Lemma 2.4. If x is in $S^i_{n-1,k-1} - K_{n-k+1}$ or is in $S^j_{n-1,k-1}$ with $i \neq j$, then $d(x) \geqslant n-2 \geqslant n-k \geqslant h$. For $x \in V(K_{n-k+1} - X)$, if exists, then $d(x) = n-1-|X| = n-h-2 \geqslant h$ for $h \leqslant \frac{n}{2}-1$. Therefore, B is an h-edge-cut of $S_{n,k}$, and so

$$\lambda_s^{(h)}(S_{n,k}) \leqslant |B| = (n-h-1)(h+1) \ \text{ for } h \leqslant \frac{n}{2} - 1.$$

If $h \geqslant \frac{n}{2}$, we choose $X = V(K^{\alpha}_{n-k+1})$. Then $|B| = (n-k+1)(k-1)$. For any x in $S^i_{n-1,k-1} - X$ or $S^j_{n-1,k-1}$ with $i \neq j$, we have $d(x) \geqslant n-2 \geqslant n-k \geqslant h$. Thus, B is an h-edge-cut of $S_{n,k}$, and so

$$\lambda_s^{(h)}(S_{n,k}) \leqslant |B| = (n-k+1)(k-1) \ \text{ for } h \geqslant \frac{n}{2}.$$

The lemma follows. ∎

Corollary 2.6 $\lambda_s^{(h)}(S_{n,2}) = n-1$ *for* $0 \leqslant h \leqslant n-2$.

Proof. On the one hand, $\lambda_s^{(h)}(S_{n,2}) \leqslant n-1$ by Lemma 2.5 when $k=2$. On the other hand, $\lambda_s^{(h)}(S_{n,2}) \geq \lambda(S_{n,2}) = n-1$ by Lemma 2.2. ∎

The following lemma shows the relations between $(n-h-1)(h+1)$ and $(n-k+1)(k-1)$.

Lemma 2.7 *For $2 \leqslant k \leqslant n-1, 0 \leqslant h \leqslant n-k$, let*

$$\psi(h,k) = \min\{(n-h-1)(h+1), (n-k+1)(k-1)\}. \tag{2.1}$$

If $h \leqslant \frac{n}{2}-1$, then

$$\psi(h,k) = \begin{cases} (n-h-1)(h+1) & \text{if } 0 \leqslant h \leqslant k-2; \\ (n-k+1)(k-1) & \text{if } h \geqslant k-1. \end{cases}$$

Proof. Let $f(x) = (n-x)\,x$, then $\psi(h,k) = \min\{f(h+1), f(k-1)\}$. It can be easily checked that $f(x)$ is a convex function on the interval $[0,n]$, the maximum value is reached at $x = \frac{n}{2}$. Thus, $f(x)$ is an increasing function on the interval $[0, \frac{n}{2}]$.

If $0 \leqslant h \leqslant k-2$, then $h+1 \leqslant k-1$. Since $h \leqslant n-k$, $h+1 \leqslant n-k+1$. Clearly, $\max\{k-1, n-k+1\} \leqslant \frac{n}{2}$. Thus, when $h \leqslant \frac{n}{2}-1$, $f(h+1) \leqslant f(k-1) = f(n-k+1)$, and so $\psi(h,k) = f(h+1) = (n-h-1)(h+1)$.

If $h \geqslant k-1$, then $k-1 < h+1 \leqslant \frac{n}{2}$. Thus, $f(k-1) < f(h+1)$, and so $\psi(n,k) = f(k-1) = (n-k+1)(k-1)$.

The lemma follows. ∎

To state and prove our main results, we need some notations. Let B be a minimum h-edge-cut of $S_{n,k}$. Clearly, $S_{n,k} - B$ has exactly two connected components. Let X and Y be two vertex-set of two connected components of $S_{n,k} - B$. For a fixed $t \in I_k \setminus \{1\}$ and any $i \in I_n$, let

$$\begin{array}{l} X_i = X \cap V(S^{t:i}_{n-1,k-1}), \\ Y_i = Y \cap V(S^{t:i}_{n-1,k-1}), \\ B_i = B \cap E(S^{t:i}_{n-1,k-1}) \text{ and} \\ B_{ij} = B \cap E(S^{t:i}_{n-1,k-1}, S^{t:j}_{n-1,k-1}), \end{array} \tag{2.2}$$

and let

$$\begin{array}{l} J = \{i \in I_n : \ X_i \neq \emptyset\}, \\ J' = \{i \in J : \ Y_i \neq \emptyset\} \ \text{ and} \\ T = \{i \in I_n : \ Y_i \neq \emptyset\}. \end{array} \tag{2.3}$$

Lemma 2.8 *Let B be a minimum h-edge-cut of $S_{n,k}$ and X be the vertex-set of a connected component of $S_{n,k} - B$. If $3 \leqslant k \leqslant n-1$ and $1 \leqslant h \leqslant n-k$ then, for any $t \in I_k \setminus \{1\}$,*

(a) *B_i is an $(h-1)$-edge-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$,*

(b) *$\lambda^{(h)}_s(S_{n,k}) \geqslant |J'|\ \lambda^{(h-1)}_s(S_{n-1,k-1})$,*

Proof. (a) By the definition of J', B_i is an edge-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$. For any vertex x in $S^{t:i}_{n-1,k-1} - B_i$, since x has degree at least h in $S_{n,k} - S$ and has exactly one neighbor outsider $S^{t:i}_{n-1,k-1}$, x has degree at least $h-1$ in $S^{t:i}_{n,k} - B_i$. This fact shows that B_i is an $(h-1)$-edge-cut of $S^{t:i}_{n-1,k-1}$ for any $i \in J'$.

(b) By the assertion (a), we have $|B_i| \geqslant \lambda^{(h-1)}_s(S_{n-1,k-1})$, and so

$$\lambda^{(h)}_s(S_{n,k}) = |B| \geqslant \sum_{i \in J'} |B_i| \geqslant |J'|\lambda^{(h-1)}_s(S_{n-1,k-1}).$$

the lemma follows. ∎

3 Proof of Theorem 1.1

Proof. By Lemma 2.5 and Lemma 2.7, we only need to prove that, for $2 \leqslant k \leqslant n-1$ and $0 \leqslant h \leqslant n-k$,

$$\lambda^{(h)}_s(S_{n,k}) \geqslant \begin{cases} (n-h-1)(h+1) & \text{for } h \leqslant k-2 \text{ and } h \leqslant \frac{n}{2}-1, \\ (n-k+1)(k-1) & \text{otherwise.} \end{cases} \tag{3.1}$$

Let $\omega(h,k)=\max\{(n-h-1)(h+1),(n-k+1)(k-1)\}$.

We proceed by induction on $k\geqslant 2$ and $h\geqslant 0$. The inequality (3.1) is true for $k=2$ and any h with $0\leqslant h\leqslant n-2$ by Corollary 2.6. The inequality (3.1) is also true for $h=0$ and any k with $2\leqslant k\leqslant n-1$ since $\lambda_s^{(0)}(S_{n,k})=\lambda(S_{n,k})=n-1$. Assume the induction hypothesis for $k-1$ with $k\geqslant 3$ and for $h-1$ with $h\geqslant 1$, that is,

$$\lambda_s^{(h-1)}(S_{n-1,k-1})\geqslant\begin{cases}(n-h)h & \text{for } h\leqslant k-3 \text{ and } h\leqslant \frac{n-1}{2},\\ (n-k+2)(k-2) & \text{otherwise.}\end{cases}\tag{3.2}$$

Let B be a minimum h-edge-cut of $S_{n,k}$ and X be the vertex-set of a minimum connected component of $S_{n,k}-B$. By Lemma 2.5, we have

$$|B|\leqslant\omega(h,k).\tag{3.3}$$

Use notations defined in (2.2) and (2.3). Choose $t\in I_k\setminus\{1\}$ such that $|J|$ is as large as possible. For each $i\in I_n$, we write $S^i_{n-1,k-1}$ for $S^{t:i}_{n-1,k-1}$ for short.

We first show $|J|=1$. Suppose to the contrary $|J|\geqslant 2$. We will deduce contradictions by considering three cases depending on $|J'|=0$, $|J'|=1$ or $|J'|\geqslant 2$.

Case 1. $|J'|=0$,

In this case, $X_i\neq\emptyset$ and $Y_i=\emptyset$ for each $i\in J$, that is, $J\cap T=\emptyset$. By $|J|\geqslant 2$ and the minimality of X, $|T|\geqslant 2$. Assume $\{i_1,i_2\}\subseteq J$ and $\{i_3,i_4\}\in T$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^{i_1}_{n-1,k-1}$ (resp. $S^{i_2}_{n-1,k-1}$) and $S^{i_3}_{n-1,k-1}$(resp. $S^{i_4}_{n-1,k-1}$), all of which are contained in B. Since $J\cap T=\emptyset$, we have that

$$|B|\geqslant 4\,\tfrac{(n-2)!}{(n-k)!}.$$

For $k=3$,

$$|B|\geqslant 4\,\tfrac{(n-2)!}{(n-k)!}\geqslant 4(n-2)>2(n-2)$$

Combining Lemma 2.5 with Lemma 2.7 yields $|B|\leqslant\lambda_s^{(h)}(S_{n,3})\leqslant 2(n-2)$, a contradiction. For $k\geqslant 4$, it is easy to check that

$$\begin{aligned}|B|&\geqslant 4\,\tfrac{(n-2)!}{(n-k)!}\geqslant 4(n-2)(n-3)=(2n-4)(2n-6)\\&>\max\{(n-h-1)(h+1),(n-k+1)(k-1)\}\\&=\omega(h,k),\end{aligned}$$

which contradicts the inequality (3.3).

Case 2. $|J'|=1$,

Without loss of generality, assume $J'=\{1\}$. By Lemma 2.8 (a), B_1 is an $(h-1)$-edge-cut of $S^1_{n-1,k-1}$.

By $|J|\geqslant 2$, there exists an $i\in J-J'$ such that $X_i=V(S^i_{n-1,k-1})$. By the minimality of X, there exists some $j\in T-J'$ such that $Y_j=V(S^j_{n-1,k-1})$. By Lemma 2.4, there are $\frac{(n-2)!}{(n-k)!}$ independent swap-edges between $S^i_{n-1,k-1}$ and $S^j_{n-1,k-1}$, thus $|B_{ij}|=\frac{(n-2)!}{(n-k)!}\geqslant n-2$. By (3.2), we consider the following two cases.

If $\lambda_s^{(h-1)}(S_{n-1,k-1})\geqslant(n-h)h$, then

$$\begin{aligned}|B|&\geqslant|B_1|+|B_{ij}|\\&\geqslant(n-h)h+(n-2)\\&>(n-h-1)h+(n-h-1)\\&=(n-h-1)(h+1),\end{aligned}$$

If $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-k+2)(k-2)$, then

$$\begin{aligned}|B| &\geqslant |B_1| + |B_{ij}| \\ &\geqslant (n-k+2)(k-2) + (n-2) \\ &> (n-k+2)(k-2) + (n-k+2) \\ &> (n-k+1)(k-1).\end{aligned}$$

Therefore, we have $|B| > \omega(h,k)$, which contradicts the inequality (3.3).

Case 3. $|J'| \geqslant 2$.

By Lemma 2.8 (b) and (3.2), we consider the following two cases.

If $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-h)h$, then

$$\begin{aligned}|B| &\geqslant |J'|\lambda_s^{(h-1)}(S_{n-1,k-1}) \\ &\geqslant 2(n-h)h \geqslant (n-h)h + (n-h) \\ &> (n-h-1)(h+1),\end{aligned}$$

If $\lambda_s^{(h-1)}(S_{n-1,k-1}) \geqslant (n-k+2)(k-2)$, then

$$\begin{aligned}|B| &\geqslant |J'|\lambda_s^{(h-1)}(S_{n-1,k-1}) \\ &\geqslant 2(n-k+2)(k-2) \\ &\geqslant (n-k+2)(k-2) + (n-k+2) \\ &> (n-k+1)(k-1).\end{aligned}$$

Therefore, we have $|B| > \omega(h,k)$, which contradicts the inequality (3.3).

Thus, we have $|J| = 1$. By the choice of t, the i-th bits of all vertices in X are same for each $i = 2, 3, \ldots, k$, and so X is a complete graph. Thus, we have that

$$\lambda_s^{(h)}(S_{n,k}) \quad = |B| = (n-|X|)|X|$$

Since $h+1 \leqslant |X| \leqslant n-k+1$ and $f(x) = (n-x)\,x$ is a convex function on the interval $[0,n]$, we have that

$$\lambda_s^{(h)}(S_{n,k}) \quad = |B| = (n-|X|)|X| \geqslant \psi(h,k),$$

where $\psi(h,k)$ is defined in (2.1).

If $h \leqslant \frac{n}{2} - 1$, using Lemma 2.7, we have

$$\lambda_s^{(h)}(S_{n,k}) \geqslant \psi(h,k) = \begin{cases} (n-h-1)(h+1) & \text{if } 0 \leqslant h \leqslant k-2; \\ (n-k+1)(k-1) & \text{if } h \geqslant k-1. \end{cases} \tag{3.4}$$

If $h \geqslant \frac{n}{2}$, we have $X = V(K_{n-k+1})$. Otherwise, there exists some $x \in V(K_{n-k+1} - X)$ such that

$$h \leqslant d(x) = n-1-|X| \leqslant n-h-2,$$

which implies $h \leqslant \frac{n}{2} - 1$, a contradiction. Therefore, we have $|X| = n-k+1$, and

$$\lambda_s^{(h)}(S_{n,k}) = |B| = (n-|X|)|X| = (n-k+1)(k-1) \text{ for } h \geqslant \tfrac{n}{2}. \tag{3.5}$$

Combining (3.4) with (3.5) yields (3.1). By the induction principle, the theorem follows. ∎

As we have known, when $k = n-1$, $S_{n,n-1}$ is isomorphic to the star graph S_n. Akers and Krishnamurthy [1] determined $\lambda(S_n)$ and $\lambda_s^{(1)}(S_n)$, which can be obtained from our result by setting $k = n-1$ and $h = 0, 1$, respectively.

Corollary 3.1 (Akers and Krishnamurthy [1]) $\lambda(S_n) = n-1$ *for* $n \geqslant 2$ *and* $\lambda_s^{(1)}(S_n) = 2n-4$ *for* $n \geqslant 3$.

References

[1] S. B. Akers and B. Krishnamurthy, A group theoretic model for symmetric interconnection networks. *IEEE Transactions on Computers*, **38** (4) (1989), 555-566.

[2] E. Cheng, M. J. Lipman, Unidirectional (n, k)-star graphs. *Journal Interconnection Networks*, **3**(1-2) (2002), 19-34.

[3] E. Cheng, M. J. Lipman, Increasing the connectivity of the star graphs. *Networks*, **40**(3) (2002), 165-169.

[4] W.-K. Chiang and R.-J. Chen, The (n, k)-star graphs: A generalized star graph. *Information Processing Letters*, **56** (1995), 259-264.

[5] J.-M. Xu, *Combinatoral Theory in Networks* (in Chinese). Academic Press, Beijing, 2007.